



Fuzzy hypervector spaces based on fuzzy singletons

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ARTICLE INFO

Article history:

Received 19 November 2010

Received in revised form 27 February 2011

Accepted 28 February 2011

Keywords:

Fuzzy hypervector space

Fuzzy singleton

Fuzzy spanning

Fuzzy freeness

ABSTRACT

We introduce and study fuzzy hypervector spaces based on fuzzy singletons. In this regard by considering the notion of fuzzy singletons, we characterize a fuzzy hypervector space fuzzily spanned by a fuzzy subset. Then we use these results to introduce the concept of fuzzy freeness of a fuzzy subset μ of a hypervector space V and finally we characterize it in terms of linear independence in the usual sense.

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1. Introduction

The notion of a hypergroup was introduced by Marty in 1934 [1]. Since then many researchers have worked on hyperalgebraic structures and developed this theory (for more see [2–4]). In 1990, Tallini introduced the notion of hypervector spaces [5,6] and studied basic properties of them.

The concept of a fuzzy subset of a nonempty set was introduced by Zadeh in 1965 [7] as a function from a nonempty set X into the unit real interval $I = [0, 1]$. Rosenfeld [8] applied this to the theory of groups and then many researchers developed it in all the fields of algebra. The concepts of a fuzzy field and a fuzzy linear space over a fuzzy field were introduced and discussed by Nanda [9]. In 1977, Katsaras and Liu [10] formulated and studied the notion of fuzzy vector subspaces over the field of real or complex numbers. Fuzzy vector spaces have been studied by Malik and Mordeson [11], Mordeson [12] studied the generating properties of fuzzy algebraic structures and Kumar [13].

Recently fuzzy set theory has been well-developed in the context of hyperalgebraic structure theory (for example see [14–21,24]). Ameri in [14] introduced and studied the notion of fuzzy hypervector space over valued fields. The authors in [22] introduced and studied fuzzy basis of fuzzy hypervector spaces. In this paper we follow [14,22] to generalize the results in [11] to fuzzy hypervector spaces. In this regard first we define a fuzzy hypervector space fuzzily spanned by a fuzzy subset θ of a hypervector space V and denote it by $\langle \theta \rangle$. Then we characterize $\langle \theta \rangle$ and use it to introduce the concept of fuzzy freeness of a fuzzy subset μ of V and characterize it in terms of linear independence in the usual sense (Theorem 4.4). We also study the notion of a fuzzy basis for μ (Theorems 4.7 and 4.10).

2. Preliminaries

In this section we present some definitions and simple properties of hypervector spaces and fuzzy subsets, that we shall use later on.

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A map $\circ : H \times H \longrightarrow P_*(H)$ is called a *hyperoperation* or join operation, where $P_*(H)$ is the set of all non-empty subsets of H . The join operation is extended to subsets of H in natural way, so that $A \circ B$ is given by

$$A \circ B = \bigcup \{a \circ b : a \in A \text{ and } b \in B\}.$$

The notations $a \circ A$ and $A \circ a$ are used for $\{a\} \circ A$ and $A \circ \{a\}$ respectively. Generally, the singleton $\{a\}$ is identified by its element a .

Definition 2.1 ([5]). Let K be a field and $(V, +)$ be an abelian group. We define a hypervector space over K to be the quadrupled $(V, +, \circ, K)$, where “ \circ ” is a mapping

$$\circ : K \times V \longrightarrow P_*(V),$$

such that for all $a, b \in K$ and $x, y \in V$ the following conditions hold:

- (H₁) $a \circ (x + y) \subseteq a \circ x + a \circ y$,
- (H₂) $(a + b) \circ x \subseteq a \circ x + b \circ x$,
- (H₃) $a \circ (b \circ x) = (ab) \circ x$,
- (H₄) $a \circ (-x) = (-a) \circ x = -(a \circ x)$,
- (H₅) $x \in 1 \circ x$.

Remark 2.2. (i) In the right-hand side of the right distributive law (H₁) the sum is meant in the sense of Frobenius, that is we consider the set of all sums of an element of $a \circ x$ with an element of $a \circ y$. Similarly we have in left distributive law (H₂).

(ii) We say that $(V, +, \circ, K)$ is *anti-left distributive* if

$$\forall a, b \in K, \forall x \in V, \quad (a + b) \circ x \supseteq a \circ x + b \circ x,$$

and *strongly left distributive*, if

$$\forall a, b \in K, \forall x \in V, \quad (a + b) \circ x = a \circ x + b \circ x.$$

In a similar way we define the *anti-right distributive* and *strongly right distributive* hypervector spaces, respectively. V is called *strongly distributive* if it is both strongly left and strongly right distributive.

(iii) The left-hand side of associative law (H₃) means the set-theoretical union of all the sets $a \circ y$, where y runs over the set $b \circ x$, i.e.

$$a \circ (b \circ x) = \bigcup_{y \in b \circ x} a \circ y.$$

(iv) Let $\Omega_V = 0 \circ 0$, where 0 is the zero of $(V, +)$. In [5] it is shown if V is either strongly right or left distributive, then Ω is a subgroup of $(V, +)$.

Example 2.3. In $(\mathbb{R}^2, +)$ we define the product times a scalar in \mathbb{R} by setting:

$$\forall a \in \mathbb{R}, \quad \forall x \in \mathbb{R}^2 : a \circ x = \begin{cases} \text{line } ox & \text{if } x \neq \underline{0}, \\ \{\underline{0}\} & \text{if } x = \underline{0}, \end{cases}$$

where $\underline{0} = (0, 0)$. Then $(\mathbb{R}^2, +, \circ, \mathbb{R})$ is a strongly left distributive hypervector space.

In the sequel of this note, unless otherwise specified, we assume that V is a hypervector space over the field K .

Definition 2.4. A nonempty subset W of V is a subhypervector space if W is itself a hypervector space with the hyperoperation on V , i.e.

$$\begin{cases} W \neq \emptyset, \\ \forall x, y \in W \implies x - y \in W, \\ \forall a \in K, \quad \forall x \in W \implies a \circ x \subseteq W. \end{cases}$$

In this case we write $W \leq V$.

Definition 2.5. If S is a nonempty subset of V , then the linear span of S is the smallest subhyperspace of V containing S , i.e.

$$\langle S \rangle = \bigcap_{S \subseteq W \leq V} W.$$

Lemma 2.6 ([15]). If S is a nonempty subset of V , then

$$\langle S \rangle = \left\{ t \in \sum_{i=1}^n a_i \circ s_i, a_i \in K, s_i \in S, n \in \mathbb{N} \right\}.$$

Definition 2.7 ([23]). A subset S of V is called *linearly independent* if for every vector v_1, v_2, \dots, v_n in S , and $c_1, \dots, c_n \in K, \underline{0} \in c_1 \circ v_1 + \dots + c_n \circ v_n$, implies that $c_1 = c_2 = \dots = c_n = 0$. A subset S of V is called *linearly dependent* if it is not

linearly independent. A basis for V is a linearly independent subset of V such that span V . We say that V has *finite dimensional* if it has a finite basis.

Definition 2.8. A hypervector space V over K is said to be K -invertible or shortly invertible if and only if $u \in a \circ v$ implies that $v \in a^{-1} \circ u$.

Theorem 2.9 ([23]). Let V be invertible. Then for every v_1, \dots, v_n in V , either v_1, \dots, v_n are linearly independent or for some $1 \leq j \leq n$, v_j is in a linear combination of the others.

Theorem 2.10 ([23]). Let V be strongly left distributive and invertible. If V is finite dimensional, then every linearly independent subset of V is contained in a finite basis.

Definition 2.11. (i) For a fuzzy subset μ of X , the level subset μ_t is defined by

$$\mu_t = \{x \in X : \mu(x) \geq t\}, \quad t \in [0, 1].$$

(ii) The image of μ is denoted by $\text{Im}(\mu)$ and is defined by

$$\text{Im}(\mu) = \{\mu(x) : x \in X\},$$

and μ is called finite-valued if $|\text{Im}(\mu)| < \infty$.

(iii) If $\mu \in \text{FS}(X)$ and $A \subseteq X$, then

$$\bar{\mu}(A) = \bigvee_{a \in A} \mu(a) \quad \text{and} \quad \underline{\mu}(A) = \bigwedge_{a \in A} \mu(a).$$

Definition 2.12 (Extension Principle). Let $f : X \longrightarrow Y$ be a mapping and $\mu \in \text{FS}(X)$ and $\nu \in \text{FS}(Y)$. Then we define $f(\mu) \in \text{FS}(Y)$ and $f^{-1}(\nu) \in \text{FS}(X)$ respectively as follows:

$$f(\mu)(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

and $f^{-1}(\nu)(x) = \nu(f(x))$, $\forall x \in X$.

Definition 2.13. Let K be a field and $\nu \in \text{FS}(K)$. Suppose the following conditions hold:

- (i) $\nu(a + b) \geq \nu(a) \wedge \nu(b)$, $\forall a, b \in K$,
- (ii) $\nu(-a) \geq \nu(a)$, $\forall a \in K$,
- (iii) $\nu(ab) \geq \nu(a) \wedge \nu(b)$, $\forall a, b \in K$,
- (iv) $\nu(a^{-1}) \geq \nu(a)$, $\forall a \in K \setminus \{0\}$.

Then we call ν a fuzzy field in K and denote it by ν_K .

Obviously, Definition 2.13 is a generalization of the classical field notion.

Definition 2.14 ([14]). Let V be a hypervector space over a field K and ν be a fuzzy field of K . A fuzzy set μ of V is said to be a fuzzy hypervector space of V over fuzzy field ν_K , if for all $x, y \in V$ and all $a \in K$, the following conditions are satisfied:

- (i) $\mu(x + y) \geq \mu(x) \wedge \mu(y)$,
- (ii) $\mu(-x) \geq \mu(x)$,
- (iii) $\bigwedge_{y \in a \circ x} \mu(y) \geq \nu(a) \wedge \mu(x)$,
- (iv) $\nu(1) \geq \mu(0)$.

Obviously, Definition 2.14 is a generalization of the concept of a fuzzy vector space and also of the classical notion of a hypervector space (in sense of [5]). If we consider $\nu = \chi_K$, the characteristic function of K , then μ is called a fuzzy subhypervector space of V .

Theorem 2.15 ([14]). A hypervector space V is finite dimensional if and only if every fuzzy hypervector space μ of V is finite-valued.

Proposition 2.16 ([22]). If μ is a fuzzy hypervector space of V over fuzzy field ν_K , then $\bar{\mu}(V) = \mu(0)$.

Example 2.17. In Example 2.3, set

$$W = \{(b, 0) : b \in \mathbb{R}\}.$$

Obviously, $(W, +, \circ, \mathbb{R})$ is a subhypervector space of $V = (\mathbb{R}^2, +, \circ, \mathbb{R})$, such that $\forall a, b \in \mathbb{R}$,

$$a \circ (b, 0) = \begin{cases} W & \text{if } b \neq 0, \\ \{0\} & \text{if } b = 0. \end{cases}$$

Choose numbers $t_1, t_2 \in [0, 1]$ such that $t_1 > t_2$. Define the fuzzy subset μ of V by

$$\mu(x) = \begin{cases} t_1 & \text{if } x \in W, \\ t_2 & \text{otherwise.} \end{cases}$$

Then μ is a fuzzy subhyperspace of V .

3. Fuzzy Spanning

Definition 3.1. Let μ, μ_1, \dots, μ_n be fuzzy subhyperspaces of V and ν be a fuzzy field of K . We define $\mu_1 + \dots + \mu_n$ and $\nu \circ \mu$ to be the fuzzy hypervector spaces of V and $P_*(V)$, respectively, whose membership functions are given by:

$$(\mu_1 + \dots + \mu_n)(x) = \bigvee \left\{ \bigwedge \{\mu_1(x_1), \dots, \mu_n(x_n)\} : x = \sum_{i=1}^n x_i, x_i \in V \right\},$$

and

$$(\nu \circ \mu)(W) = \bigvee \{(\nu(a) \wedge \mu(t)) : a \in K, t \in V, W \subseteq a \circ t\}.$$

Remark 3.2. If μ, μ_1, \dots, μ_n are fuzzy subhyperspaces of V and ν is a fuzzy field of K and $a \in K$, then

- (i) $(\nu \circ \mu)(x) = (\nu \circ \mu)(\{x\}) = \bigvee \{(\nu(a) \wedge \mu(t)) : a \in K, t \in V, x \in a \circ t\},$
- (ii) $(a \circ \mu)(W) = (\chi_a \circ \mu)(W) = \bigvee_{W \subseteq a \circ t} \mu(t),$
- (iii) $(a \circ \mu)(x) = (a \circ \mu)(\{x\}) = \bigvee_{x \in a \circ t} \mu(t).$

Lemma 3.3. Let V be invertible and strongly left distributive and let μ be a fuzzy hypervector space of V over ν_K . Then

- (i) for $a \neq 0, \forall x \in V : (a \circ \mu)(x) \geq \nu(a) \wedge \mu(x),$
- (ii) for $a = 0, (a \circ \mu)(x) = \bar{\mu}(V).$

Proof. (i) If $a \neq 0$, then by Remark 3.2, and Definitions 2.8, 2.13 and 2.14 we have:

$$\begin{aligned} (a \circ \mu)(x) &= \bigvee_{x \in a \circ t} \mu(t) \\ &= \bigvee_{t \in a^{-1} \circ x} \mu(t) \\ &\geq \bigwedge_{t \in a^{-1} \circ x} \mu(t) \\ &\geq \nu(a^{-1}) \wedge \mu(x) \\ &\geq \nu(a) \wedge \mu(x). \end{aligned}$$

(ii) If $a = 0$, since $0 \in 0 \circ x$, then by Proposition 2.16, it is concluded that:

$$(a \circ \mu)(x) = \bigvee_{x \in 0 \circ t} \mu(t) = \mu(0) = \bar{\mu}(V). \quad \square$$

Definition 3.4. Let X be a set, $A \subseteq X$ and $0 \leq \alpha \leq 1$. Then the fuzzy subset A_α of X is defined by:

$$A_\alpha(x) = \begin{cases} \alpha & x \in A, \\ 0 & x \notin A. \end{cases}$$

Also for every $x \in X$,

$$x_\alpha(\hat{x}) = \begin{cases} \alpha & \hat{x} = x, \\ 0 & \hat{x} \neq x. \end{cases}$$

The fuzzy subset A_α is called a fuzzy singleton.

Proposition 3.5. Let V be invertible and strongly left distributive and μ be a fuzzy hypervector space of V over fuzzy field ν_K . Suppose that $y \in V$, $a \in K$ and $0 \leq \alpha, \beta \leq 1$. Then for all $x \in V$, the following statements hold:

- (i) If $a \neq 0$, then $(a_\alpha \circ \mu)(x) = \alpha \wedge \bar{\mu}(a^{-1} \circ x) \geq \alpha \wedge \nu(a) \wedge \mu(x)$.
 (ii) $(0_\alpha \circ \mu)(x) = \begin{cases} \alpha \wedge \bar{\mu}(V) & \text{if } x = \underline{0}, \\ 0 & \text{if } x \neq \underline{0}. \end{cases}$
 (iii) $(\nu \circ y_\beta)(x) = \begin{cases} \bigvee \{(\nu(b) \wedge \beta) : b \in K, x \in b \circ y\} & \text{if } y \neq \underline{0} \text{ and } x \in \langle y \rangle, \\ 0 & \text{if } y \neq \underline{0} \text{ and } x \notin \langle y \rangle. \end{cases}$
 (iv) If V is strongly right distributive, then

$$(\nu \circ \underline{0}_\beta)(x) = \begin{cases} \bar{\nu}(K) \wedge \beta & \text{if } x = \underline{0}, \\ 0 & \text{if } x \neq \underline{0}. \end{cases}$$

Proof. (i)

$$\begin{aligned} (a_\alpha \circ \mu)(x) &= \bigvee \{(a_\alpha(b) \wedge \mu(t)) : b \in K, t \in V, x \in b \circ t\} \\ &= \bigvee \{(\alpha \wedge \mu(t)) : t \in V, x \in a \circ t\} \\ &= \bigvee \{(\alpha \wedge \mu(t)) : t \in V, t \in a^{-1} \circ x\} \\ &= \bigvee \{(\alpha \wedge \mu(t)) : t \in a^{-1} \circ x\} \\ &= \alpha \wedge \left(\bigvee_{t \in a^{-1} \circ x} \mu(t) \right) \\ &= \alpha \wedge \bar{\mu}(a^{-1} \circ x) \\ &\geq \alpha \wedge \underline{\mu}(a^{-1} \circ x) \\ &\geq \alpha \wedge \nu(a^{-1}) \wedge \mu(x) \\ &\geq \alpha \wedge \nu(a) \wedge \mu(x). \end{aligned}$$

(ii)

$$\begin{aligned} (0_\alpha \circ \mu)(\underline{0}) &= \bigvee \{(0_\alpha(b) \wedge \mu(t)) : b \in K, t \in V, \underline{0} \in b \circ t\} \\ &= \bigvee \{(\alpha \wedge \mu(t)) : t \in V, \underline{0} \in \underline{0} \circ t\} \\ &= \bigvee \{(\alpha \wedge \mu(t)) : t \in V\} \\ &= \alpha \wedge \left(\bigvee_{t \in V} \mu(t) \right) \\ &= \alpha \wedge \bar{\mu}(V), \end{aligned}$$

since $\underline{0} \circ t$ is a subhypervector space of V . If $x \neq \underline{0}$, then $0_\alpha(b) = 0$, since $b \neq 0$ when $\underline{0} \notin b \circ t$. So $(0_\alpha \circ \mu)(x) = 0$.

(iii) If $x \in \langle y \rangle$, then

$$\begin{aligned} (\nu \circ y_\beta)(x) &= \bigvee \{(\nu(b) \wedge y_\beta(t)) : b \in K, t \in V, x \in b \circ t\} \\ &= \bigvee \{(\nu(b) \wedge \beta) : b \in K, x \in b \circ y\}. \end{aligned}$$

(iv) If $x = \underline{0}$, then

$$\begin{aligned} (\nu \circ \underline{0}_\beta)(x) &= \bigvee \{(\nu(b) \wedge \underline{0}_\beta(t)) : b \in K, t \in V, x \in b \circ t\} \\ &= \bigvee \{(\nu(b) \wedge \beta) : b \in K, t \in V, \underline{0} \in b \circ \underline{0}\} \\ &= \bigvee \{(\nu(b) \wedge \beta) : b \in K\} \\ &= \left(\bigvee_{b \in K} \nu(b) \right) \wedge \beta \\ &= \bar{\nu}(K) \wedge \beta, \end{aligned}$$

since in this case $b \circ \underline{0}$ is a subhypervector space of V . If $x \neq \underline{0}$, then $\underline{0}_\beta(t) = 0$, since $t \neq \underline{0}$, when $x \in b \circ t$. \square

Proposition 3.6. Let $a, b \in K$, $x, y \in V$ and $0 \leq \alpha, \beta, \gamma, \delta \leq 1$. Then

- (i) $a_\alpha \circ x_\beta = (a \circ x)_{\alpha \wedge \beta}$,
 (ii) $x_\beta + y_\delta = (x + y)_{\beta \wedge \delta}$,
 (iii) $a_\alpha \circ x_\beta + b_\gamma \circ y_\delta = (a \circ x + b \circ y)_{\alpha \wedge \beta \wedge \gamma \wedge \delta}$.

Proof. (i)

$$\begin{aligned}(a_\alpha \circ x_\beta)(z) &= \bigvee \{(a_\alpha(c) \wedge x_\beta(v)) : c \in K, v \in V, z \in c \circ v\} \\ &= \begin{cases} \bigvee (\alpha \wedge \beta) = \alpha \wedge \beta & \text{if } a = c, x = v, \\ 0 & \text{otherwise,} \end{cases} \\ &= (a \circ x)_{\alpha \wedge \beta}(z).\end{aligned}$$

(ii)

$$\begin{aligned}(x_\beta + y_\delta)(z) &= \bigvee \{(x_\beta(v) \wedge y_\delta(w)) : v, w \in V, z = v + w\} \\ &= \begin{cases} \bigvee (\beta \wedge \delta) = \beta \wedge \delta & \text{if } x = v, y = w, \\ 0 & \text{otherwise,} \end{cases} \\ &= (x + y)_{\beta \wedge \delta}(z).\end{aligned}$$

(iii) The result follows from conditions (i) and (ii). \square

Remark 3.7. If $a_{1\alpha_1}, \dots, a_{n\alpha_n}$ and $x_{1\beta_1}, \dots, x_{n\beta_n}$ are fuzzy singletons, where $a_i \in K$ and $x_i \in V, i = 1, \dots, n$, then $\sum_{i=1}^n a_{i\alpha_i} \circ x_{i\beta_i}$ is called a fuzzy linear combination of fuzzy singletons. By Proposition 3.6, it follows that a fuzzy linear combination of fuzzy singletons is a fuzzy singleton in V .

Proposition 3.8. Let V be invertible and strongly left distributive such that $|1 \circ x| = 1$, for every $x \in V$. Let μ be a fuzzy hypervector space of V over the fuzzy field v_K and let η, θ be fuzzy subsets of V such that $\eta, \theta \subseteq \mu$, and $a, b \in K$. Then $a_\alpha \circ \eta + b_\beta \circ \theta \subseteq \mu$, where $0 \leq \alpha \leq v(a)$ and $0 \leq \beta \leq v(b)$.

Proof. Since $a_\alpha \circ \eta$ and $b_\beta \circ \theta$ are fuzzy subsets of V , it suffices to show that $a_\alpha \circ \eta \subseteq \mu$ and $\eta + \theta \subseteq \mu$. Suppose that $a \neq 0$. Let $z \in V$. Then

$$\begin{aligned}(a_\alpha \circ \eta)(z) &= \bigvee \{(a_\alpha(b) \wedge \eta(t)) : b \in K, t \in V, z \in b \circ t\} \\ &= \bigvee \{\alpha \wedge \eta(t) : t \in V, z \in a \circ t\} \\ &= \bigvee \{\alpha \wedge \eta(t) : t \in a^{-1} \circ z\} \\ &\leq \bigvee \{v(a) \wedge \mu(t) : t \in a^{-1} \circ z\} \\ &\leq \bigvee \left(\bigwedge_{s \in a \circ t} \mu(s) : t \in a^{-1} \circ z \right) \\ &= \bigvee_{t \in a^{-1} \circ z} \left(\bigwedge_{s \in a \circ t} \mu(s) \right) \\ &\leq \bigvee_{t \in a^{-1} \circ z} \left(\bigvee_{s \in a \circ t} \mu(s) \right) \\ &\leq \bigvee_{s \in a \circ (a^{-1} \circ z)} \mu(s) \\ &= \bigvee_{s \in (aa^{-1}) \circ z} \mu(s) \\ &= \bigvee_{s \in 1 \circ z} \mu(s) \\ &= \bar{\mu}(1 \circ z) \\ &= \mu(z),\end{aligned}$$

by Definitions 2.1 and 2.14. Suppose that $a = 0$ and $z = \underline{0}$. Then

$$\begin{aligned}(a_\alpha \circ \eta)(\underline{0}) &= \alpha \wedge \bar{\mu}(V) \\ &= \alpha \wedge \mu(\underline{0}) \\ &\leq v(0) \wedge \mu(\underline{0}) \\ &\leq \mu(\underline{0}),\end{aligned}$$

by **Propositions 2.16** and **3.5**. Now

$$\begin{aligned}(\eta + \theta)(z) &= \bigvee \{(\eta(v) \wedge \theta(w)) : v, w \in V, z = v + w\} \\ &\leq \bigvee \{(\mu(v) \wedge \mu(w)) : v, w \in V, z = v + w\} \\ &\leq \mu(z). \quad \square\end{aligned}$$

Definition 3.9. Let μ be a fuzzy hypervector space of V over the fuzzy field v_K , and let θ be a fuzzy subset of V such that $\theta \subseteq \mu$. Let $\langle \theta \rangle$ denote the intersection of all fuzzy hypervector spaces of V over the fuzzy field v_K , that contain θ and are contained in μ . Then $\langle \theta \rangle$ is called the fuzzy subhyperspace of μ fuzzily spanned (or generated) by θ .

Theorem 3.10. Let V be invertible and strongly left distributive such that $|1 \circ x| = 1$, for every $x \in V$. Let μ be a fuzzy hypervector space of V over the fuzzy field v_K , and let θ be a fuzzy subset of V such that $\theta \subseteq \mu$. Define the fuzzy subset η of V by the following:

$$\eta(x) = \bigvee \left\{ \left(\sum_{i=1}^n a_{i\alpha_i} \circ x_{i\beta_i} \right) (x) : a_i \in K, x_i \in V, v(a_i) = \alpha_i, \theta(x_i) = \beta_i, i = 1, \dots, n, n \geq 1 \right\}.$$

Then η is a fuzzy hypervector space of V over the fuzzy field v_K and $\langle \theta \rangle = \eta$.

Proof. We have $x_{i\beta_i} \subseteq \theta \subseteq \langle \theta \rangle$. Thus by **Propositions 3.6** and **3.8**, $\eta \subseteq \langle \theta \rangle$. In order to show that $\eta \supseteq \langle \theta \rangle$, it suffices to show that η is a fuzzy hypervector space of V over the fuzzy field v_K and $\eta \supseteq \theta$. Let $x \in V$ and let $\theta(x) = \beta$. Then $\eta(x) \geq x_\beta(x)$ and so $\eta \supseteq x_\beta$. Thus $\eta \supseteq \theta$. Let $u, v \in V$. Then $\eta(u)$ and $\eta(v)$ are supremums of the numbers of the forms, $(\sum_{i=1}^p b_{i\gamma_i} \circ y_{i\delta_i})(u)$ and $(\sum_{i=1}^q c_{i\kappa_i} \circ z_{i\lambda_i})(v)$, respectively. Suppose that $\eta(u) > 0$ and $\eta(v) > 0$. Then there exist sequences

$$\gamma_j^* = \gamma_{1j} \wedge \dots \wedge \gamma_{pj} \wedge \delta_{1j} \wedge \dots \wedge \delta_{pj} \quad \text{and} \quad \kappa_j^* = \kappa_{1j} \wedge \dots \wedge \kappa_{qj} \wedge \lambda_{1j} \wedge \dots \wedge \lambda_{qj},$$

such that $\gamma_j^* \rightarrow \eta(u)$ and $\kappa_j^* \rightarrow \eta(v)$ (as the limit of sequences γ_j^*, κ_j^* in the unite real interval $[0,1]$). Now if $u \in \langle y_1, \dots, y_p \rangle$ and $v \in \langle z_1, \dots, z_q \rangle$, then $u + v \in \langle y_1, \dots, y_p, z_1, \dots, z_q \rangle$. Thus for $j = 1, 2, \dots$,

$$\begin{aligned}\eta(u + v) &\geq \bigwedge \{ \gamma_{ij}, \delta_{ij}, \kappa_{kj}, \lambda_{kj} : i = 1, \dots, p, k = 1, \dots, q \} \\ &= \gamma_j^* \wedge \kappa_j^*.\end{aligned}$$

Since $\gamma_j^* \wedge \kappa_j^* \rightarrow \eta(u) \wedge \eta(v)$, so

$$\eta(u + v) \geq \eta(u) \wedge \eta(v).$$

If either $\eta(u) = 0$ or $\eta(v) = 0$, then clearly $\eta(u + v) \geq \eta(u) \wedge \eta(v)$. Clearly $\eta(-x) = \eta(x)$ for all $x \in V$, by (H_4) . Let $a \in K, x \in V$ and $\beta = \alpha_1 \wedge \dots \wedge \alpha_n \wedge \beta_1 \wedge \dots \wedge \beta_n$. Suppose $a \neq 0$. Now since V is invertible, so for $t \in a \circ x$, $t \in \sum_{i=1}^n a_i \circ x_i$ if and only if $x \in \sum_{i=1}^n (a^{-1}a_i) \circ x_i$, and we have:

$$\begin{aligned}\left(\sum_{i=1}^n a_{i\alpha_i} \circ x_{i\beta_i} \right) (t) &= \beta = \alpha_1 \wedge \dots \wedge \alpha_n \wedge \beta_1 \wedge \dots \wedge \beta_n \\ &= v(a_1) \wedge \dots \wedge v(a_n) \wedge \beta_1 \wedge \dots \wedge \beta_n \\ &\geq v(a) \wedge v(a_1) \wedge \dots \wedge v(a_n) \wedge \beta_1 \wedge \dots \wedge \beta_n \\ &\geq v(a) \wedge v(a^{-1}a_1) \wedge \dots \wedge v(a^{-1}a_n) \wedge \beta_1 \wedge \dots \wedge \beta_n.\end{aligned}$$

Thus for all $t \in a \circ x$,

$$\begin{aligned}\eta(t) &= \bigvee \left\{ \left(\sum_{i=1}^n a_{i\alpha_i} \circ x_{i\beta_i} \right) (t) : t \in \sum_{i=1}^n a_i \circ x_i \right\} \\ &= \bigvee \left\{ v(a_1) \wedge \dots \wedge v(a_n) \wedge \beta_1 \wedge \dots \wedge \beta_n : t \in \sum_{i=1}^n a_i \circ x_i \right\} \\ &\geq \bigvee \left\{ v(a) \wedge v(a^{-1}a_1) \wedge \dots \wedge v(a^{-1}a_n) \wedge \beta_1 \wedge \dots \wedge \beta_n : x \in \sum_{i=1}^n (a^{-1}a_i) \circ x_i \right\} \\ &\geq v(a) \wedge \bigvee \left\{ v(a^{-1}a_1) \wedge \dots \wedge v(a^{-1}a_n) \wedge \beta_1 \wedge \dots \wedge \beta_n : x \in \sum_{i=1}^n (a^{-1}a_i) \circ x_i \right\} \\ &= v(a) \wedge \eta(x).\end{aligned}$$

Therefore

$$\bigwedge_{t \in a \circ x, a \neq 0} \eta(t) \geq v(a) \wedge \eta(x),$$

where the latter inequality holds because the number on the left-hand side of the inequality equals either $v(a)$ or $\eta(x)$. Suppose that $a = 0$ and $l \in 0 \circ x$. Then

$$\begin{aligned} \eta(l) &= \bigvee \left\{ \left(\sum_{i=1}^n a_{i\alpha_i} \circ x_{i\beta_i} \right) (l) : a_i \in K, x_i \in V, v(a_i) = \alpha_i, \theta(x_i) = \beta_i, i = 1, \dots, n, n \geq 1 \right\} \\ &\geq \bigvee \{ (0_\alpha \circ y_\beta)(l) : 0 \in K, y \in V, v(0) = \alpha = 1, \theta(y) = \beta \} \\ &= \bigvee \{ 1 \wedge \beta : y \in V, \theta(y) = \beta \} \\ &= \bigvee \{ \theta(y) : y \in V \} \\ &\geq \eta(x) \\ &\geq v(0) \wedge \eta(x). \end{aligned}$$

Thus

$$\bigwedge_{l \in 0 \circ x} \eta(l) \geq v(0) \wedge \eta(x). \quad \square$$

4. Fuzzy freeness

Let θ be a fuzzy subset of V and let \mathcal{S} be a set of fuzzy singletons of V such that if $x_\alpha, x_\beta \in \mathcal{S}$, then $\alpha = \beta > 0$. Define the fuzzy subset $\theta(\mathcal{S})$ of V by the following:

$$\theta(\mathcal{S})(x) = \begin{cases} \alpha & \text{if } x_\alpha \in \mathcal{S}, \\ 0 & \text{otherwise.} \end{cases}$$

Define $\langle \mathcal{S} \rangle = \langle \theta(\mathcal{S}) \rangle$. Let η be a fuzzy subset of V . Define

$$\mathcal{S}(\eta) = \{x_\alpha : x \in V, \alpha = \eta(x) > 0\}.$$

Then it follows that:

$$\eta(\mathcal{S}(\eta)) = \eta \quad \text{and} \quad \mathcal{S}(\eta(\mathcal{S})) = \mathcal{S}.$$

If there are only a finite number of $x_\alpha \in \mathcal{S}$ with $\alpha > 0$, we call \mathcal{S} finite. If $\eta(x) > 0$ for only a finite number of $x \in V$, we call η finite. Clearly \mathcal{S} is finite if and only if $\eta(\mathcal{S})$ is finite and η is finite if and only if $\mathcal{S}(\eta)$ is finite. For $x \in V$, let $\eta \setminus x$ denote the fuzzy subset of V defined by the following:

$$(\eta \setminus x)(y) = \begin{cases} \eta(y) & \text{if } y \neq x, \\ 0 & \text{if } y = x. \end{cases}$$

Definition 4.1. Let μ be a fuzzy hypervector space of V over the fuzzy field v_K , and let θ be a fuzzy subset of V such that $\theta \subseteq \mu$. Then θ is called a fuzzy system of generators of μ over v_K if $\langle \theta \rangle = \mu$, and θ is said to be fuzzy free over v_K if for all $x_\alpha \subseteq \theta$, where $\alpha = \theta(x)$, $x_\alpha \not\subseteq \langle \theta \setminus x \rangle$. θ is said to be a fuzzy basis for μ if θ is a fuzzy system of generators of μ and θ is fuzzy free. Let \mathcal{S} denote a set of fuzzy singletons of V such that if $x_\alpha, x_\beta \in \mathcal{S}$, then $\alpha = \beta$ and $x_\alpha \subseteq \mu$. Then \mathcal{S} is called a fuzzy singleton system of generators of μ over v_K if $\langle \mathcal{S} \rangle = \mu$. \mathcal{S} is said to be fuzzy free over v_K if for all $x_\alpha \in \mathcal{S}$, $x_\alpha \not\subseteq \langle \mathcal{S} \setminus \{x_\alpha\} \rangle$. \mathcal{S} is said to be a fuzzy basis of singletons for μ if \mathcal{S} is a fuzzy singleton system of generators of μ and \mathcal{S} is fuzzy free.

Remark 4.2. If \mathcal{S} is a set of fuzzy singletons of V such that either x_0 or $0_\alpha \in \mathcal{S}$, then \mathcal{S} is not fuzzy free over v_K , because $x_0 \subseteq \langle \mathcal{S} \rangle$ or $0_\alpha \subseteq \langle \mathcal{S} \rangle$.

Proposition 4.3. Let μ be a nonzero fuzzy hypervector space of V over the fuzzy field v_K . Set

$$V^* = \{x \in V : \mu(x) > 0\} \quad \text{and} \quad K^* = \{a \in K : v(a) > 0\}.$$

Then

- (i) K^* is a subfield of K ;
- (ii) V^* is a subhypervector space of V over K^* .

Proof. (i) Since $v(0) = v(1) = 1 > 0$, so $0, 1 \in K^*$. Thus $K \setminus \{0\} \neq \emptyset$. Now for all $a, b \in K^*$, by Definition 2.13 we have:

$$\begin{aligned} v(a - b) &\geq v(a) \wedge v(-b) \geq v(a) \wedge v(b) > 0, \\ v(ab) &\geq v(a) \wedge v(b) > 0, \\ v(a^{-1}) &\geq v(a) > 0, \quad \text{for } a \neq 0. \end{aligned}$$

Therefore $a - b, ab, a^{-1} \in K^*$.

(ii) Since μ is nonzero, so $\mu(\underline{0}) > 0$ and thus $\underline{0} \in V^*$. Hence $V^* \neq \emptyset$. Now for all $x, y \in V^*$ and for all $a \in K^*$, we have:

$$\begin{aligned} \mu(x - y) &\geq \mu(x) \wedge \mu(y) > 0, \\ \mu(a \circ x) &\geq v(a) \wedge \mu(x) > 0. \end{aligned}$$

Therefore $x - y \in V^*$ and $a \circ x \in V^*$. \square

Theorem 4.4. Let V be invertible and strongly left distributive such that $|1 \circ x| = 1$, for every $x \in V$. Let μ be a fuzzy hypervector space of V over the fuzzy field v_K and let

$$\mathcal{S} \subseteq \{x_\alpha : x \in V^*, 0 < \alpha \leq \mu(x)\},$$

such that if $x_\alpha, x_\beta \in \mathcal{S}$, then $\alpha = \beta$ and let $X = \{x : x_\alpha \in \mathcal{S}\}$. Suppose that $v_K(K) \geq \bar{\mu}(V \setminus \{\underline{0}\})$. Then \mathcal{S} is fuzzy free over v_K if and only if X is linearly independent over K .

Proof. Suppose X is not linearly independent over K . If $\underline{0} \in X$, then $\underline{0}_\alpha \in \mathcal{S}$ and so \mathcal{S} is not fuzzy free over v_K . Thus suppose $\underline{0} \notin X$. Then by Theorem 2.9, there exists $x \in X, x_1, \dots, x_n \in X$, and $a_1, \dots, a_n \in K$ such that $x \in \sum_{i=1}^n a_i \circ x_i$, where $a_i \neq 0, i = 1, \dots, n$. Suppose that $x_\beta \subseteq \sum_{i=1}^n a_{i\alpha_i} \circ x_{i\beta_i}$, where $\beta = \mu(\mathcal{S})(x), \alpha_i = v(a_i)$ and $\beta_i = \mu(\mathcal{S})(x_i), i = 1, \dots, n$. Then $x_\beta \subseteq \langle \mathcal{S} \setminus \{x_\beta\} \rangle$ by Theorem 3.10. Thus by Remark 4.2, \mathcal{S} is not fuzzy free over v_K . Suppose that $x_\beta \not\subseteq \sum_{i=1}^n a_{i\alpha_i} \circ x_{i\beta_i}$, i.e.

$$\begin{aligned} \beta &> \alpha_1 \wedge \dots \wedge \alpha_n \wedge \beta_1 \wedge \dots \wedge \beta_n \\ &\geq \beta_1 \wedge \dots \wedge \beta_n. \end{aligned}$$

Let $\beta_1 = \beta_1 \wedge \dots \wedge \beta_n$. Then by invertibility of V we have:

$$x_1 \in \sum_{i=2}^n (-a_i a_1^{-1}) \circ x_i + a_1^{-1} \circ x,$$

and

$$x_{1\beta_1} \subseteq \sum_{i=2}^n (-a_i a_1^{-1})_{\gamma_i} \circ x_{i\beta_i} + (a_1^{-1})_{\alpha_1} \circ x_\beta \subseteq \langle \mathcal{S} \setminus \{x_{1\beta_1}\} \rangle,$$

where $\gamma_i = v(-a_i a_1^{-1}), i = 2, \dots, n$. Thus \mathcal{S} is not fuzzy free over v_K . Conversely, let \mathcal{S} be not fuzzy free over v_K . Then there exists $x_\beta \in \mathcal{S}$ such that $x_\beta \subseteq \langle \mathcal{S} \setminus \{x_\beta\} \rangle$. Hence by Theorem 3.10 there exists $a_{ij} \in K, x_{ij} \in X \setminus \{x\}$ such that $x \in \sum_{i=1}^{n_j} a_{ij} \circ x_{ij}$ and

$$\min\{\alpha_{1j}, \dots, \alpha_{n_j}, \beta_{1j}, \dots, \beta_{n_j}\} \rightarrow \beta^* \geq \beta,$$

where $\beta = \mu(\mathcal{S})(x) > 0, \langle \mathcal{S} \setminus \{x_\beta\} \rangle(x) = \beta^*, \alpha_{ij} = v(a_{ij})$, and $\beta_{ij} = \mu(\mathcal{S})(x_{ij}), i = 1, \dots, n$. Thus $x \in \langle x_{1j}, \dots, x_{n_j} \rangle \subseteq \langle X \setminus \{x\} \rangle$. Therefore X is not linearly independent over K . \square

Example 4.5. In abelian group $(\mathbb{C}, +)$ we define the external hyperoperation $\circ : \mathbb{R} \times \mathbb{C} \longrightarrow P_*(\mathbb{C})$ by

$$a \circ x = \{z \in \mathbb{C} : |z| \leq |a| |x|\}.$$

Then it is easy to verify that $V = (\mathbb{C}, +, \circ, \mathbb{R})$ is a hypervector space. Define the fuzzy subset $v_{\mathbb{R}}$ of \mathbb{R} by

$$v_{\mathbb{R}}(a) = \begin{cases} 1 & a \in \mathbb{Q}, \\ 1/2 & a \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Clearly $v_{\mathbb{R}}$ is a fuzzy subfield of \mathbb{R} . Let μ be the fuzzy subset of V defined by

$$\mu(x) = \begin{cases} 3/4 & x = 1, i, \\ 0 & \text{otherwise.} \end{cases}$$

Then $i_{3/4} \neq 1_{3/4} \circ i_{1/2}$ and $1_{3/4} \neq (1)_{1/2} \circ i_{3/4}$. Thus it follows that $i_{3/4} \not\subseteq \langle 1_{3/4} \rangle$ and $1_{3/4} \not\subseteq \langle i_{3/4} \rangle$. Hence $\mathcal{S} = \{1_{3/4}, i_{3/4}\}$ is fuzzy free over $v_{\mathbb{R}}$. However, $X = \{1, i\}$ is not linearly independent over \mathbb{R} . Note that $v_{\mathbb{R}}(\mathbb{R}) < \bar{\mu}(V \setminus \{\underline{0}\})$.

Definition 4.6. Let μ be a fuzzy hypervector space of V over the fuzzy field v_K and let \mathcal{S} be a set of fuzzy singletons of V such that if $x_\alpha, x_\beta \in \mathcal{S}$, then $\alpha = \beta$ and $x_\alpha \subseteq \mu$. Then \mathcal{S} is said to be maximally fuzzy free in μ over v_K if \mathcal{S} is fuzzy free over v_K and there does not exist a fuzzy singleton y_α such that $y_\alpha \subseteq \mu$ and $\mathcal{S} \cup \{y_\alpha\}$ is fuzzy free over v_K , where $y \in V$. A fuzzy subset θ of V is said to be maximally fuzzy free in μ over v_K if $\theta \subseteq \mu$, θ is fuzzy free over v_K , and there does not exist a fuzzy free subset η of V such that $\theta \subseteq \eta \subseteq \mu$ and $\theta = \eta \setminus y$ for some $y \in V$ with $\eta(y) > 0$.

Theorem 4.7. Let V be invertible and strongly left distributive such that $|1 \circ x| = 1$, for every $x \in V$. Let μ be a nonzero fuzzy hypervector space of V over the fuzzy field v_K and let

$$\mathcal{S} \subseteq \{x_\alpha : x \in V^*, 0 < \alpha \leq \mu(x)\}$$

be such that if $x_\alpha, x_\beta \in \mathcal{S}$, then $\alpha = \beta$ and let $X = \{x : x_\alpha \in \mathcal{S}\}$. Suppose that $v_K(K) \geq \bar{\mu}(V \setminus \{0\})$. Then \mathcal{S} is maximally fuzzy free in μ over v_K if and only if X is a basis for V^* over K^* .

Proof. Since μ is nonzero, so $V^* \supset \{0\}$, and thus $K^* = K$. Suppose \mathcal{S} is maximal fuzzy free. Then by Theorem 4.4, X is linearly independent over K . Suppose $\langle X \rangle \supset V^*$. Then there exists $x \in V^* \setminus \langle X \rangle$. Hence $X \cup \{x\}$ is linearly independent. Clearly $x_\alpha \not\subseteq \mathcal{S}$, where $\alpha = \mu(x)$, since $x \notin X$. Now $\alpha > 0$, since $x \in V^*$. By Theorem 4.4, $\mathcal{S} \cup \{x_\alpha\}$ is fuzzy free in μ . Hence \mathcal{S} is not maximal, a contradiction. Therefore $\langle X \rangle = V^*$. Conversely, suppose that X is a basis of V^* over K . Then by Theorem 4.4 \mathcal{S} is fuzzy free in μ over v_K . Suppose that \mathcal{S} is not maximal. Then there exists $x \in V^*$ such that $\mathcal{S} \cup \{x_\alpha\}$, $\alpha \leq \mu(x)$, is fuzzy free over v_K . Now $x \notin X$ and by Theorem 4.4, $X \cup \{x\}$ is linearly independent over K , a contradiction. Therefore \mathcal{S} is maximal. \square

Corollary 4.8. Let V be invertible and strongly left distributive. Let μ be a fuzzy hypervector space of V over the fuzzy field v_K such that $v_K(K) \geq \bar{\mu}(V \setminus \{0\})$. Then μ has maximally fuzzy free sets over v_K of fuzzy singletons of V and every such set has the same cardinality.

Let μ be a fuzzy hypervector space of V over the fuzzy field v_K . We now show that a maximal fuzzy free set of singletons of μ need not fuzzily generate μ .

Example 4.9. Let $V = (K^2, +, \cdot, K)$. Define the fuzzy subset θ of V by

$$\theta(p, q) = \begin{cases} 1 & (p, q) = (1, 1), \\ 1/2 & (p, q) = (1, 0), (0, 1), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mu = \langle \theta \rangle$ and $v_K = \chi_K$. Then by Theorem 4.4, $\mathcal{S} = \{(1, 0)_{1/2}, (0, 1)_{1/2}\}$ is a maximally fuzzy free set of fuzzy singletons in μ . However, \mathcal{S} is not a fuzzy basis for μ over v_K , because $(1, 1)_1 \not\subseteq \langle (1, 0)_{1/2}, (0, 1)_{1/2} \rangle$.

Note that even though $(1, 1)_1 \not\subseteq \langle (1, 0)_{1/2}, (0, 1)_{1/2} \rangle$, $\{(1, 0)_{1/2}, (0, 1)_{1/2}, (1, 1)_1\}$ is not fuzzy free. $\{(1, 1)_1, (0, 1)_{1/2}\}$ is a fuzzy basis of μ over v_K .

Theorem 4.10. Let V be finite dimensional and let μ be a fuzzy hypervector space of V over the fuzzy field v_K such that $v_K(K) \geq \bar{\mu}(V \setminus \{0\})$. If μ is finite valued, then μ has a fuzzy basis over v_K .

Proof. Since V is finite dimensional, so by Theorem 2.15, μ is finite valued. Thus assume that $\text{Im}(\mu) = \{t_1, \dots, t_n\}$, where $t_1 < \dots < t_n = 1$. Then $V = \mu_{t_1} \supset \dots \supset \mu_{t_n}$. Construct a basis β for V as follows: Let β_n be a basis for subhypervector space μ_{t_n} , and by Theorem 2.10, β_n has been extended to a basis $\beta_n \cup \dots \cup \beta_{i+1}$ for $\mu_{t_{i+1}}$. Extend $\beta_n \cup \dots \cup \beta_{i+1}$ to a basis $\beta_n \cup \dots \cup \beta_{i+1} \cup \beta_i$ for μ_{t_i} , $i = n, \dots, 1$. Put

$$\beta = \beta_n \cup \dots \cup \beta_1.$$

Now if $x \in V$, then $\mu(x) = t_i$, for some i , and so $x \in \mu_{t_i} = \langle \beta_n \cup \dots \cup \beta_i \rangle$. Let

$$\mathcal{S} = \begin{cases} \{x_{t_m} : x \in \beta \cap \mu_{t_m}, m = 1, \dots, n\}, & t_1 > 0, \\ \{x_{t_m} : x \in \beta \cap \mu_{t_m}, m = 2, \dots, n\}, & t_1 = 0. \end{cases}$$

Then $x_{t_i} \subseteq \langle \mathcal{S} \rangle$, because $x \in \langle \beta_n \cup \dots \cup \beta_i \rangle$ and $t_i < t_{i+1} < \dots < t_n$. In fact, $\langle \mathcal{S} \rangle(x) = t_i$, i.e. $\langle \mathcal{S} \rangle(x) = \mu(x)$. Hence $\langle \mathcal{S} \rangle = \mu$. Therefore by Theorem 4.4, \mathcal{S} is fuzzy free over v_K . \square

The next example shows the difference between subhypervector spaces and fuzzy subhyperspaces. In [23] it was proved that if V is strongly left distributive and invertible, then every non-independentless spanning subset of V contains a basis.

Example 4.11. In the ring $(\mathbb{R}, +, \cdot)$ we define the external hyperoperation $\circ : \mathbb{R} \times \mathbb{R} \rightarrow P_*(\mathbb{R})$ by $a \circ b = \{-ab, ab\}$. It is easy to verify that $V = (\mathbb{R}, +, \circ, \mathbb{R})$ is a strongly left distributive and invertible hypervector space. Define a fuzzy subset θ of V by

$$\theta(x) = \begin{cases} \frac{k-1}{k} & x = \frac{k-1}{k}, k = 2, 3, \dots, \\ 0 & \text{otherwise.} \end{cases}$$

Then for $x \in V$,

$$\left(\left(\frac{xk}{k-1} \right)_1 \circ \left(\frac{k-1}{k} \right)_{\frac{k-1}{k}} \right) (x) = \left(\frac{xk}{k-1} \circ \frac{k-1}{k} \right)_{\frac{k-1}{k}} (x) = \{-x, x\}_{\frac{k-1}{k}} (x) = \frac{k-1}{k}.$$

Thus by Theorem 3.10, for $v_K = \chi_K$ and $\mu = \chi_V$, $\langle \theta \rangle(x) = 1$. That is, θ is a fuzzy system of generators of μ , but θ does not contain a fuzzy basis of μ (since for example $\left(\frac{1}{2}\right)\left(\frac{1}{3}\right) \subseteq \left(\eta \setminus \frac{1}{2}\right)$, for any subset η of θ).

Corollary 4.12. *Let V be invertible and strongly left distributive and let μ be a fuzzy hypervector space of V over the fuzzy field v_K such that $v_K(K) \geq \bar{\mu}(V \setminus \{0\})$. If μ is finitely fuzzily generated over v_K , then μ has a fuzzy basis over v_K .*

Proof. Since μ is finitely fuzzily generated, thus there exists a fuzzy subset θ of V such that $\mu = \langle \theta \rangle$ and $\mu(x) > 0$ for only finitely many $x \in V$. Hence the supremum in Theorem 3.10, can be replaced by maximum. Therefore $\text{Im}(\mu) \subseteq \{\theta(x) : x \in V\}$. The desired result holds from Theorem 4.10. \square

5. Conclusion

We introduced and studied the basic properties of fuzzy hypervector spaces based on fuzzy singletons. We obtained a characterization of fuzzy hypervector spaces fuzzily spanned by a fuzzy subset. Finally we introduced fuzzy freeness of a fuzzy subset of a given hypervector space and investigate its basic properties.

Acknowledgements

This research is partially supported by the “Fuzzy Systems and Its Applications Center of Excellence, Shahid Bahonar University of Kerman, Iran” and “Research Center in Algebraic Hyperstructures and Fuzzy Mathematics, University of Mazandaran, Babolsar, Iran”.

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